# ON THE ELLIPTIC VARIATIONAL INEQUALITY OF THE FIRST KIND FOR THE OBSTACLE PROBLEM AND IT'S APPROXIMATION.


**Sahar Muhsen Jaabar**
*Dept. of Math.,*
*College of Education,*
*Babylon University*

**Fai'z Ali Rashid**
*Dept. of Computer College of science for female*
*Babylon University*



## Abstract

The elliptic variational inequality of the first kind for the "obstacle problem" is considered. This elliptic variational inequality is related to second order partial differential operator. The physical and mathematical interpretation and some properties of the solution are given.


## 1- Introduction

An important and very useful class of non-linear problems arising from mechanics, Physics etc. consists of the so-called variational inequalities. In this paper we shall restrict our attention to the study of the existence, uniqueness, and properties of the solutions of elliptic variational inequality EVI, which have two classes, namely EVI of the first kind EVI of the second kind.

### 1-1: Notations:

- V: real Hilbert space with scalar product (. , . ) and associated norm $\|\cdot\|$.
- $V^*$: The dual space of V.
- a(. , .): $V \times V \to \Re$ is a bilinear, continuous and V- elliptic mapping on $V \times V$ .

  A bilinear form a(. , .) is said to be V-elliptic if there exists a positive constant

$\alpha$ such that $a(v,v) \geq \|v\|^2 \qquad \forall\, v \in V$ .

In general we do not assume a(.,.) to be symmetric, since in some applications non-symmetric bilinear forms may occur naturally [1]:

- $L : V \to \Re$ continuous , linear functional.
- K : is a closed, convex, non-empty subset of V.
- $j(.) : V \to \overline{\Re} = \Re \cup \{\infty\}$ is a convex, lower semi-continuous (L.S.C) and proper functional.

(j(.) is proper if <j(v)>->∞ ∀ v∈V and j≠∞).

### 1.2: EVI of First Kind

To find u∈V such that u is a solution of the problem :

$$P_1 \ldots \begin{cases} a(u, v-u) \geq L(v-u) \quad , \qquad \forall v \in K \\ u \in K \end{cases}$$

### 1.3: EVI of Second Kind

To find u∈V such that u is a solution of the problem :

$$P_2 \ldots\ldots \begin{cases} a(u, v-u) + j(v) - j(u) \geq L(v-u) & , \forall \ v \in V \\ u \in V \end{cases}$$

**1.4: Existence and Uniqueness Results for EVI of first Kind**
**1.4.1: A Theorem of Existence and Uniqueness**

Theorem 1.4.1 [2]: The problem $P_1$ has one and only one solution.

***Proof: 1- Uniqueness***

Let $u_1$ and $u_2$ be solutions of ($P_1$).We have then:

$$a(u_1, v-u_1) \geq L(v-u_1) \qquad \forall\, v \in K, u_1 \in K \ \ldots\ldots\ldots(1)$$

$$a(u_2, v-u_2) \geq L(v-u_2) \qquad \forall\, v \in K, u_2 \in K \ \ldots\ldots(2)$$

putting $u_2$ for v in (1) and $u_1$ for v in (2) and adding we get, by using the V-ellipticity of a(.,.),

$$\alpha \|u_2 - u_1\|^2 \leq a(u_2 - u_1, u_2 - u_1) \leq 0$$

which implies $u_1=u_2$, since $\alpha>0$.

**2. Existence:** We will reduce the problem ($P_1$) to a fixed point problem. By the Riesz representation theorem for Hilbert spaces there exist $A\in\ (V,V)$ ($A=A^t$ if a(.,.) is symmetric) and $\ell \in V$ such that:

(Au,v)= a(u,v) $\quad \forall u,v\in V$

and L(v)=( $\ell$ ,v) $\quad \forall v\in V$ ……………..(3)

Then the problem ($P_1$) is equivalent to finding $u\in V$ such that:

$$\begin{cases} (u - \rho(Au - \ell) - u, v - u) \leq 0 & \forall v \in K \\ u \in K & , \rho > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4) \end{cases}$$

This is equivalent to finding u such that

$$\begin{cases} (u - \rho(Au - \ell) - u, v - u) \leq 0 & \forall v \in K \\ u \in K & , \rho > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4) \end{cases}$$

This is equivalent to finding u such that:

$u = P_k(u - \rho(Au - \ell))$ , for some $\rho > 0$, ……………..(5)

where $P_k$ denotes the projection operator from V to K in the $\|\cdot\|_v$ consider the *map* $W_p(v)=P_k(v-\rho(Av-\ell))$…………………..(6)

Let $v_1, v_2 \in V$, then since $P_k$ is a contraction we have:

$$\begin{cases} \|W_p(v_1) - W_p(v_2)\|^2 \leq \|v_2 - v_1\|^2 + \\ \rho^2 \|A(v_2 - v_1)\|^2 - 2\rho a(v_2 - v_1, v_2 - v_1) \end{cases}$$

Hence we have

$$\|W_p(v_1) - W\rho(v_2)\|^2 \leq (1 - 2\rho\alpha + \rho^2\|A\|^2)\|v_2 - v_1\|^2 \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Thus $W\rho$ is a strict contraction mapping if $0 < \rho < \frac{2\alpha}{\|A\|^2}$. By taking $\rho$ in this range we have a unique solution for the fixed point problem which implies the existence of a solution for ($P_1$).

## 2- An Example of EVI of the First Kind "The Obstacle Problem"

### 2.1: Notations

* $\Omega$ : a bounded domain in $\Re^2$.
* $\Gamma$: $\partial\Omega$.
* $x=\{x_1, x_2\}$ a generic point of $\Omega$.
* $\nabla = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\}$
* $C^m(\overline{\Omega})$: space of m-times continuously differentiable real valued functions for which all the derivative up to order m are continuous on $\overline{\Omega}$.
* $C_0^m(\Omega) = \{v \in C^m(\overline{\Omega})$: supp (v) is a compact subset of $\Omega\}$
* $\|v\|_{m,p,\Omega} = \sum_{|\alpha| \le m} \|D^\alpha v\|_{L^p(\Omega)}$ for $v \in C^m(\overline{\Omega})$ where $\alpha=(\alpha_1, \alpha_2)$ ; $\alpha_1$, $\alpha$ non-negative integers, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial X_1^{\alpha_1} \partial X_2^{\alpha_2}}$
* $W^{m,p}(\Omega)$: completion of $C^m(\overline{\Omega})$ in the norm defined above.
* $W_0^{m,p}(\Omega)$: completion of $C_0^m(\Omega)$ in the above norm.
* $H^m(\Omega) = W^{m,2}(\Omega)$,
* $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

**2.2: The Mathematical Interpretation of The Problem** :

Let $V = H_0^1(\Omega) = \{V \in H^1(\Omega) : V|_\Gamma$ = trace of V on $\Gamma$=0}

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v$$

where:

$$\nabla u.\nabla v = \frac{\partial u}{\partial x_1} \cdot \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial v}{\partial x_,},$$

*L(v)=<f,v> for* $f \in V^* = H^{-1}(\Omega)$ and $v \in V$.

Let $\psi \in H^1(\Omega) \cap C^0(\overline{\Omega})$ *and* $\Psi|_\Gamma \le 0$

Define $k = \{v \in H_0^1(\Omega) : v \ge \Psi \ a.e. on \, \Omega\}$

Then the obstacle problem is a($P_1$) problem defined by:

Find u such that:

$$(2.2)\text{..........} \begin{cases} a(u, v-u) \ge L(v-u) & \forall v \in k \\ u \in k \end{cases}$$

**2.3: The Physical Interpretation of The Problem**

Let an elastic membrane occupy a region Ω in the $x_1$, $x_2$ plane, this membrane is fixed along the boundary Γ of Ω. When these is no obstacle, from the theory of elasticity the vertical displacement u, obtained by applying a vertical force F, is given by the Dirichlet problem.

$$(2.3)\ldots\ldots\ldots \begin{cases} -\Delta u = f & in\,\Omega \\ u|_{\Gamma} = 0 \end{cases}$$

Where f= F/t, t being the tension.

Where there is an obstacle, we have a free boundary problem and the displacement u satisfies the variational inequality (2.2) with Ψ being the height of the obstacle.

Similar EVI also occur, sometimes with non-symmetric bilinear forms, in mathematical models for the following problems:-

- Lubrication phenomena [3].
- Filtration of liquids in porous media [1].
- Two dimensional, irrotational flows of perfect fluids [4], [5], [6].

## 3. Existence and Uniqueness Results of The "Obstacle Problem"

For proving the existence and uniqueness of the problem (2.2) in section 2.2, we need the following lemmas stated below without proof [2]:-

LEMMA (1): Let Ω be a bounded domain in $\Re^N$. Then the semi-norm on $H^1(\Omega)$

$$v \rightarrow \left( \int_{\Omega} |\nabla v|^2 dx \right)^{1/2}$$

is a norm on $H_0^1(\Omega)$ and it is equivalent to the norm on $H_0^1(\Omega)$ induced from $H^1(\Omega)$

- The above lemma is known as Poincare-Friedrichs lemma.

**LEMMA (2):** Let $f = \Re \rightarrow \Re$ be uniformly Lipschitz continuous (i. e. ∃ k>0 such that $|f(t) - f(t') \leq k(t - t') \quad t_1 t' \in \Re$) and such $f'$ has a finite number of points of discontinuity. Then the induced map $f^*$ on $H^1(\Omega)$ defined by $u \rightarrow f(u)$ is a continuous map in to $H^1(\Omega)$. Similar results holds for $H_0^1(\Omega)$ whenever f(0)=0.

**Corollary**

If $V^*$ and $V^-$ denote the positive and the negative parts of V for $v \in H^1(\Omega)$ (respectively $H_0^1(\Omega)$) then the map $v \rightarrow \{V^+, V^-\}$ is continuous from $H^1(\Omega) \rightarrow H^1(\Omega) \times H^1(\Omega)$ (respectively $H_0^1(\Omega) \rightarrow H_0^1(\Omega) \times H_0^1(\Omega)$). Also $v \rightarrow |v|$ is continuous.

**3.1: Proof the Existence and Uniqueness of the Problem (2.2)**

In order to apply theorem (1.4.1), we have to prove that a(.,.) is V-elliptic and that k is a closed, convex, non-empty set.

The V-ellipticity of a(.,.) follows from Lemma (1) and the convexity of k is trivial , then:

**1- k is non-empty**

we have: $\Psi \in H^1(\Omega) \cap C^0(\overline{\Omega})$ with $\Psi \le 0$ on Γ. Hence, by the above corollary, $\psi^+ \in H^1(\Omega)$. Since $\Psi\big|_\Gamma \le 0$ we have $\Psi^+ \le 0\big|$. This implies $\Psi^+ \in H_0^1(\Omega)$, then $\Psi^+ = \max\{\Psi, 0\} \ge \Psi$.

Thus $\Psi^+ \in k$. Hence k is non-empty.

**2- k is closed**

Let $v_n \to v$ strongly in $H_0^1(\Omega)$ where $v_n \in k$ and $v \in H_0^1(\Omega)$. Hence $v_n \to v$ strongly in $L_2(\Omega)$. Therefore we can extract a subsequence $\{v_{ni}\}$ such that $v_{ni} \to v$ a. e. on Ω. Then $v_{ni} \ge \Psi$ a. e. on Ω implies that: $v \ge \Psi$ a. e. on Ω; therefore v∈k. Hence, by The. (1.4.1), we have a unique solution for (2.2).

## 4. Interpretation of the Problem (2.2) as a Free Boundary Problem

For the solution u of (2.2) we define:

$$\Omega^+ = \{x : x \in \Omega, \quad u(x) > \Psi(x)\}$$

$$\Omega^0 = \{x : x \in \Omega, \quad u(x) = \Psi(x)\},$$

$$\gamma = \partial\Omega^+ \cap \partial\Omega^0; u^+ = u\big|_{\Omega^+}; u^0 = u\big|_{\Omega^0}$$

We can formulate the problem (2.2) as the problem of finding γ (the free boundary) and u such that:

$$-\Delta u = f \qquad on\ \Omega^+, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.1)$$

$$u = \Psi \qquad on\ \Omega^0, \ldots\ldots\ldots\ldots\ldots\ldots(4.2)$$

$$u = 0 \qquad on\ \Gamma, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.3)$$

$$u^+\big|_\gamma = u^0\big|_\gamma \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.4)$$

The physical interpretation of these relations is the following: (4.1) means That on $\Omega^+$ the membrane is strictly over the obstacle, (4.2) means that on $\Omega^0$ the membrane is in contact with the obstacle, (4.4) is a transmission relation at the free boundary.

Actually (4.1)-(4.4) are not sufficient to characterize u since there are an infinity of solutions for (4.1)-(4.4). therefore it is necessary to add other transmission properties: for instance, if Ψ is smooth enough (say $\Psi \in H^2(\Omega)$), we require the continuity of $\nabla u$ at γ

$$(\nabla u \in H^1(\Omega) \times H^1(\Omega)).$$

This kind of free boundary interpretation holds for several problems modeled by EVI of first kind and second kind.

## 5. Regularity of Solution

we state without proof the following regularity theorem for the problem (2.2).

Theorem 5.1: (BREZIS-STAMPACCHIA [7]):

Let Ω be a bounded domain in $\Re^2$ with a smooth boundary . It

$$L(v) = \int_\Omega fv \qquad with \quad f \in L^p(\Omega), \qquad 1 < p < \infty$$

and $\Psi \in W^{2,p}(\Omega)$,

Then the solution of the problem (2.2) is in $W^{2,p}(\Omega)$

**LEMMA 5.2 [2]:** Let Ω be a bounded domain of $\Re^N$ with a boundary Γ sufficiently smooth. Then $\|\Delta v\|_{L^2(\Omega)}$ defines a norm on $H^2(\Omega) \cap H_0^1(\Omega)$ which is equivalent to the norm induced by the $H^2(\Omega)$-norm.

We shall now apply the lemma 5.2 to prove the following theorem using a method of BREZIS-STAMPACCHIA [7].

Theorem 5.3: If Γ is smooth enough, Ψ=0 and $L(v) = \int_\Omega fv$ with $f \in L^2(\Omega)$ then the solution u of the problem (2.2) satisfies:

$u \in k \cap H^2(\Omega)$,

$\|\Delta u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$ ………………………..(5.1)

<u>**Proof**</u>: From section (3.1), it follows that problem (2.2) has a unique solution u, with L and Ψ as above.

Let ∈>0, consider the following Dirichlet problem

$$\begin{cases} - \in \Delta u_\in & in\,\Omega, \\ u_\in \Gamma| = 0 \end{cases} \quad \text{..................................................(5.2)}$$

Problem (5.2) has a unique solution in $H_0^1(\Omega)$ and the smoothness of Γ assures that $u_\in$ belongs to $H^2(\Omega)$. Since u≥0 a. e. on Ω, by the maximum principle for second order elliptic differential operators [8], we have $u_\in \geq 0$. Hence:

$u_\in \in k$ ………………………………………(5.3)

from (5.3) and (2.2) we obtain:

$a(u, u_\in - u) \geq L(u_\in - u) = \int_\Omega f(u_\in - u)$……….(5.4)

The V-ellipticity of a(.,.) implies

$$a(u_\in, u_\in - u) = a(u_\in - u, u_\in - u) + a(u, u_\in - u) \geq a(u, u_\in - u)$$

so that by (5.4) we have:

$a(u_\in, u_\in - u) \geq \int_\Omega f(u_\in - u)$……………(5.5)

By (5.2) and (5.5) we obtain:

$$\in \int_\Omega \nabla u_\in \cdot \nabla(\Delta u_\in) \qquad \geq \in \int_\Omega f \Delta u_\in$$

so that

$\int_\Omega \nabla u_\in \cdot \nabla(\Delta u_\in) \geq \int_\Omega f\Delta u_\in$ ……………….(5.6)

By Green's formule , (5.6) implies

$$-\int_\Omega (\Delta u_\in)^2 dx \geq \int_\Omega f \Delta u_\in$$

Thus $\|\Delta u_\in\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$ ……………….(5.7)

Using Schwarz inequality in $L^2(\Omega)$

By Lemma (5.2) and relations (5.2), (5.7) we obtain $\lim_{\in \to 0} u_{\in} = u$ weakly in $H^2(\Omega)$, ……………………..(5.8)

(which implies that lim $u_{\in}$=u strongly in $H^{s}(\Omega)$, for every s<2),

so that u $\in H^2(\Omega)$ with

$\|\Delta u_{\in}\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$ ……………..(5.9)

## References


[1] : Comincioli V., On some oblique derivative problems arising in the fluid flow in porous media. A theoretical and numerical approach. Applied Math. And optimization, Vol.1, No4, (1975), p.313-336.

[2]: Lions J. L., Stampacchia G., Variational Inequalities , comm.. Pure Applied Math., XX, (1967). Pp.493-519.

[3]: Cryerc W., The method of christoferson for solving free boundary problems for infinite journal bearings by means of finite differences. Math. Comp. 25, (1971), pp.435-443.

[4]: Rezis H., Stampacchia G., The hodo graph method in fluid dynamics in the light of variational inequalities, Arch. Rat. Mech. Anal. 61, (1976), pp.1-18.

[5] : Brezis H., Anew method in the study of subsonic flows. In partial Differential Equations and Related Topics, J. Goldstein ed., Lecture Notes in math., Vol. 446, springier-verlag, Berlin, (1975), pp.50-60.

[6]: Framzini J. B., Finnemore E. J., Fluid Mechamics with Engineering Applications , McGraw-Hill comp. Inc., (1997).

[7]: Brezis H., Stampacchia G., The regularity of solution of variational inequalities, Bull. SocMath. France, 96, (1968). Pp. 153-180.

[8]: Asmar N. H., Partial Differential Equations and Boundary value Problems, Prentice Hall. (2000).


## الخلاصة


تم اعتبار المتباينة التغايرية الناقصية من النوع الاول "لمسألة العائق". هذه المتباينة التغايرية الناقصية تعود الى مؤثر تفاضلي جزئي من الرتبة الثانية. وقد تم برهنة التفسير الرياضي والفيزيائي وبعض خواص الحل.